ME-221
SOLUTIONS FOR PROBLEM SET 8-9

Problem 1

The impulse response of the system is:

$$g(t) = 2\varepsilon(t)e^{-2t}$$

a) The corresponding transfer function of the system is:

$$G(s) = \mathcal{L}\big[g(t)\big] = \frac{2}{s+2} \tag{1}$$

b) The unit step response ($u(t) = \varepsilon(t)$) is:

$$Y(s) = G(s)U(s) = \frac{2}{(s+2)}\frac{1}{s} = \frac{1}{s} - \frac{1}{s+2}$$

Therefore,

$$\gamma(t) = y(t) = \mathcal{L}^{-1}\big[Y(s)\big] = \varepsilon(t) - \varepsilon(t)e^{-2t} = \varepsilon(t)[1 - e^{-2t}]$$

c) We have a first order system with no zeros. Transfer function in equation 1 can be written in the following form:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{K}{\tau s + 1} \tag{2}$$

where the time constant $\tau = 0.5$ and the steady-state gain $K = 1$.

d) For all systems with a transfer function which can be written as equation 2, the time it takes for the output to reach 95% of its steady-state value is:
$t = 3\tau = 1.5$ s

e) The Laplace transform of the input $u(t) = \varepsilon(t-1)e^{-(t-1)}$ is obtained using the temporal translation property:

$$U(s) = e^{-s}\mathcal{L}\big[\varepsilon(t)e^{-t}\big] = \frac{e^{-s}}{s+1}$$

The response is:

$$y(t) = \mathcal{L}^{-1}\big[G(s)U(s)\big] = \mathcal{L}^{-1}\Big[\frac{2}{(s+2)}\frac{e^{-s}}{(s+1)}\Big] = \mathcal{L}^{-1}\big[Y_1(s)e^{-s}\big]$$

We will first consider the term $Y_1(s) = \dfrac{2}{(s+2)(s+1)}$ as the term e^{-s} will only delay the response $y_1(t)$ of a unit time.

$$Y_1(s) = \frac{2}{(s+2)(s+1)} = \frac{2}{s+1} - \frac{2}{s+2}$$

Therefore:

$$y_1(t) = 2e^{-t} - 2e^{-2t} \qquad t \geq 0$$

and:

$$y(t) = y_1(t-1) = 2e^{-(t-1)} - 2e^{-2(t-1)} \qquad t \geq 1$$

Problem 2

a) The Laplace transform of the system is:

$$Y(s)[s^2 + ks + 4] = U(s) \tag{3}$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2 + ks + 4} \tag{4}$$

It is a second-order system without zeros, which can be written in the form:

$$G(s) = K\frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

Rewriting equation 4 we get:

$$G(s) = \frac{1}{4}\frac{4}{s^2 + ks + 4} \tag{5}$$

with the steady state gain $K = 1/4$, natural frequency $\omega_0 = \sqrt{4} = 2$, $2\zeta\omega_0 = k$ and therefore damping coefficient $\zeta = k/4$.

b) The poles of the system are:

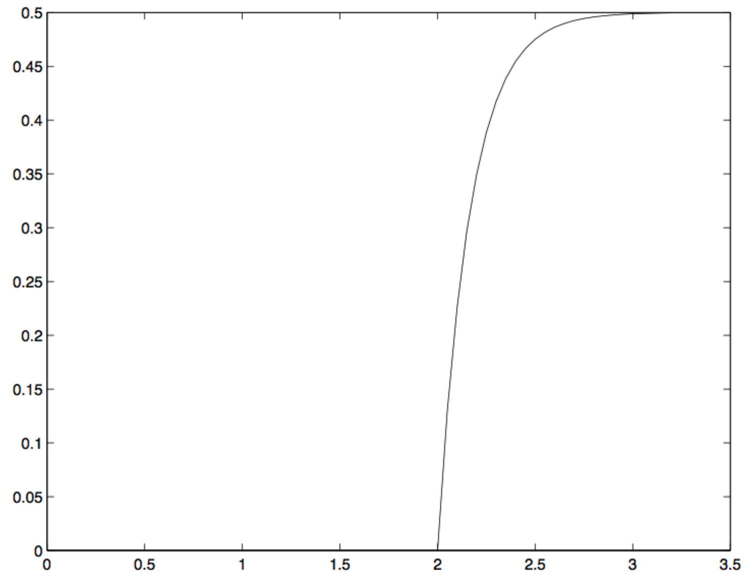
$$p_{1,2} = -\omega_0(\zeta \pm \sqrt{\zeta^2 - 1}) = -\frac{k}{2} \pm \sqrt{\frac{k^2}{4} - 4}$$

The type of the response will depend on the value of k:

- For $k < 0$, we have that $\text{Re}\{p_{1,2}\} > 0$. Because of this, the system is unstable as the response includes *increasing* exponential terms. Also, $\text{Im}\{p_{1,2}\} \neq 0$, therefore the system will be oscillatory.
- For $k = 0$, we have that $\text{Re}\{p_{1,2}\} = 0$, and $\text{Im}\{p_{1,2}\} \neq 0$, the system will be undamped. In other words, it will never converge nor diverge (because of the real part) but will keep oscillating (because of the imaginary part).
- For $0 < k < 4$, we have that $\text{Re}\{p_{1,2}\} < 0$, leading to a stable system. $\zeta < 1$, therefore, the system will be underdamped. There will be an oscillatory motion since $\text{Im}\{p_{1,2}\} \neq 0$.
- For $k = 4$, we have that $\text{Re}\{p_{1,2}\} < 0$, and $\zeta = 1$ (there will be one repeated pole: $p_1 = p_2$). Therefore, the system will be stable and critically damped. There will be no oscillatory motion since $\text{Im}\{p_{1,2}\} = 0$.
- For $k > 4$, we have that $\text{Re}\{p_{1,2}\} < 0$, and $\zeta > 1$ Therefore, the system will be stable and overdamped. There will be no oscillatory motion since $\text{Im}\{p_{1,2}\} = 0$.

Problem 3

The unit-step response is here plotted:



a) The graph shows the output response of a first-order system delayed by 2 seconds. The transfer function will hence be:

$$G(s) = \frac{Ke^{-2s}}{\tau s + 1}$$

The steady state gain K is the ratio between the final output value (0.5) and the amplitude of the input step function (1). Therefore, $K = 0.5$.
The time constant τ, corresponds to the time frame between the system's first reaction ($t = 2s$) and the moment in which the response reaches 63% of its final value ($t = 2.167s$). Hence, $\tau = 0.167s$.

$$G(s) = \frac{0.5e^{-2s}}{0.167s + 1} = \frac{3e^{-2s}}{s + 6}$$

b)

$$g(t) = \mathcal{L}^{-1}\left[G(s)\right] = 3\varepsilon(t-2)e^{-6(t-2)}$$

Problem 4

a) We know that the transfer function of the system is:

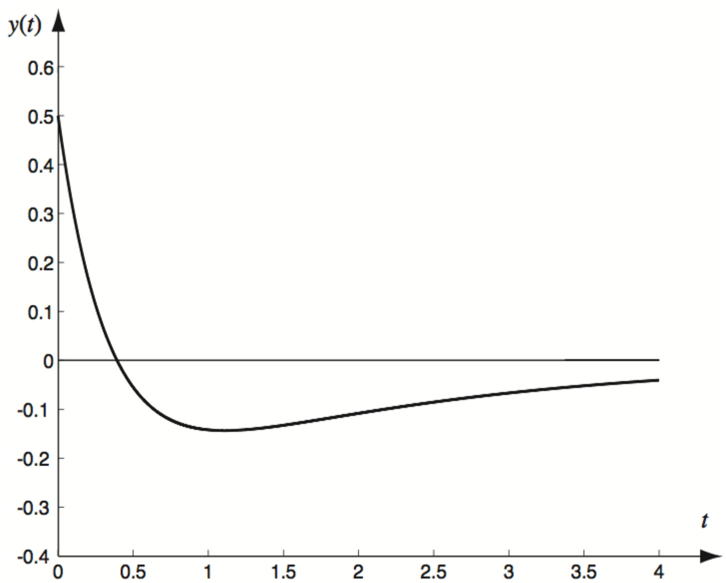
$$G(s) = \frac{s-1}{(2s+1)(s+3)}$$

The impulse response of the system is therefore:

$$Y(s) = G(s)U(s) = \frac{s-1}{(2s+1)(s+3)} = \frac{A}{2s+1} + \frac{B}{s+3}$$
$$\rightarrow A = \lim_{s\rightarrow -1/2} \frac{s-1}{s+3} = -3/5$$
$$\rightarrow B = \lim_{s\rightarrow -3} \frac{s-1}{2s+1} = 4/5$$

The impulse response is hence:

$$y(t) = \mathcal{L}^{-1}Y(s) = -\frac{3}{10}e^{-t/2} + \frac{4}{5}e^{-3t} \quad t \geq 0$$



b) $z_1 = 1$, $p_1 = -0.5$ and $p_2 = -3$. Both poles are in the left quadrants, the system is stable.

Problem 5

a) The value of the unit step response at peak time is given by:

$$\gamma(t_p) = K(1 + e^{-\zeta\pi/\sqrt{1-\zeta^2}})$$

From the plot, we can read the static gain, which is $K = 2$. The peak value is recorded as $\gamma(t_p) = 3$ in the plot. We can extract the value of the damping coefficient ζ as follows:

$$\gamma(t_p) = 2(1 + e^{-\zeta\pi/\sqrt{1-\zeta^2}}) = 3$$
$$\rightarrow \zeta = \frac{ln2}{\sqrt{\pi^2 + (ln2)^2}} = 0.215$$

We can calculate ω_0 from the value of t_0:

$$t_p = \frac{\pi}{\bar{\omega}} = \frac{\pi}{\omega_0\sqrt{1-\zeta^2}} = 1 \rightarrow \omega_0 = \frac{\pi}{\sqrt{1-\zeta^2}} = 3.217$$

b)The trasfer function is therefore:

$$G(s) = Ke^{-s}\frac{\omega^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} = \frac{20.6e^{-s}}{s^2 + 1.38s + 10.3}$$

(Note 1s delay)

Problem 6

a) Input: force $F(t)$, Output: position $x(t)$. We can then write the mathematical model of the Newton's law for $t \geq 0$:

$$(m + M)\ddot{x} = Mg - kx - f\dot{x}$$

With: $x(0) = \dot{x}(0) = 0$

Remark:
Left side of the equation$\rightarrow$ both masses $(m + M)$ are accelerated.
Right side of the equation $\rightarrow$ only M is taken into account as m is already taken into account in the definition $x = 0$ for $t \leq 0$.

b) This system is rather particular as the application of the input modifies one of parameter of the system (m for $t < 0$ and $m + M$ for $t \geq 0$). However, for $t \geq 0$, the mass of the system remains constant.

The Laplace transform of the system is:

$$[(m + M)s^2 + fs + k]X(s) = F(s)$$
$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{(m + M)s^2 + fs + k} = \frac{1}{100s^2 + 350s + 3500}$$

It is a second-order system without zeros. Its poles are:

$$100s^2 + 350s + 3500 = 2s^2 + 7s + 70 = 0$$
$$p_{1,2} = \frac{-7 \pm \sqrt{49 - 560}}{4} = -1.75 \pm 5.65j$$

c) As it is a second-order system, the denominator can be written as: $\tau^2 s^2 + 2\zeta\tau s + 1$

Hence,

$$\frac{1}{35}s^2 + \frac{1}{10}s + 1$$

With: $\tau = 0.169s$, and $\zeta = 0.269$. This corresponds to an highly oscillatory response.

d) The response is obtained by:

$$X(s) = G(s)F(s) = \frac{1}{100s^2 + 350s + 3500}(90)(9.81)\frac{1}{s}$$
$$= \frac{17.66}{s(2s^2 + 7s + 70)} = \frac{A}{s} + \frac{Bs + C}{2s^2 + 7s + 70}$$

Where: $A = 0.252$, $B = -0.504$ and $C = -1.766$.

By looking at the laplace transform tables, the denominator informs us on which form we should have it to enable us to easily find the inverse Laplace. In this case:

$$2s^2 + 7s + 70 = 2(s^2 + 3.5s + 35)$$

resembles the following denominator from the Laplace tables:

$$(s + \alpha)^2 + \omega^2 = s^2 + 2\alpha s + \alpha^2 + \omega^2$$

We therefore get:

$$2\alpha = 3.5$$
$$\alpha^2 + \omega^2 = 35$$

We find that: $\alpha = 1.75$ and $\omega = 5.65$.

The next step is to work on the numerator. With what we found above we have:

$$\frac{Bs + C}{2s^2 + 7s + 70} = \frac{\frac{B}{2}s + \frac{C}{2}}{s^2 + 3.5s + 35} = \frac{\frac{B}{2}s + \frac{C}{2}}{(s + \alpha)^2 + \omega^2} = \frac{-0.252s - 0.883}{(s + \alpha)^2 + \omega^2}$$

The numerator should also be put into a form similar to that found in the Laplace table. In this case, we can see that we can write it as a linear combination of $(s + \alpha)$ and ω:

$$-0.252s - 0.833 = c_1(s + \alpha) + c_2\omega$$

We get:

$$c_1 = -0.252$$
$$-0.883 = -0.252\alpha + c_2\omega \Rightarrow c_2 = -0.078$$

Therefore, we have:

$$\frac{Bs + C}{2s^2 + 7s + 70} = -0.252\frac{(s+\alpha)}{(s+\alpha)^2 + \omega^2} - 0.078\frac{\omega}{(s+\alpha)^2 + \omega^2}$$

The inverse Laplace transform of X(s) gives:

$$x(t) = 0.252[1 - e^{-1.75t}cos5.65t - 0.31e^{-1.75t}sin5.65t] \quad t \geq 0$$